Non-continuous interval maps and rotation theory in spiking neuron models

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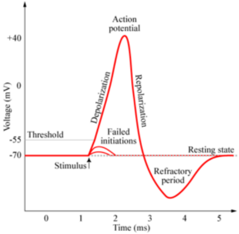
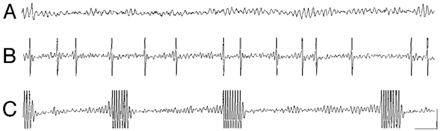
1. Neuron models
2. Geometric mechanism for mixed-mode oscillations
   - Adaptation map
3. Rotation theory
   - Non-overlapping case
   - Overlapping case
   - Some intermediate cases
4. Conclusions

What characterizes neurons' activity? Neurons are electrically excitable cells that communicate through the emission of action potentials (spikes): stereotyped membrane potential electrical impulses. They encode information in the way spikes are emitted through:

- the answer to specific simple stimuli (excitability properties)
- and the spike pattern fired,
- often related to properties of the interspike behavior (e.g.: sub-threshold oscillations) .





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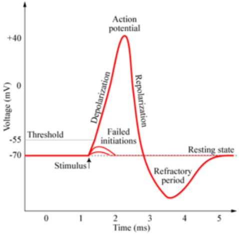

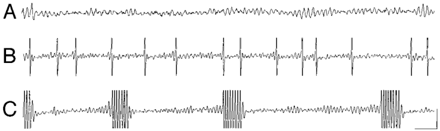

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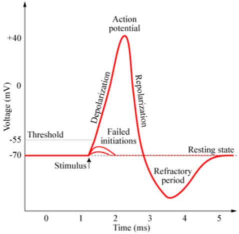
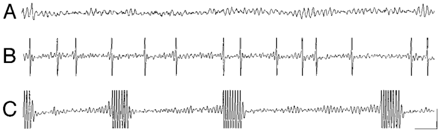
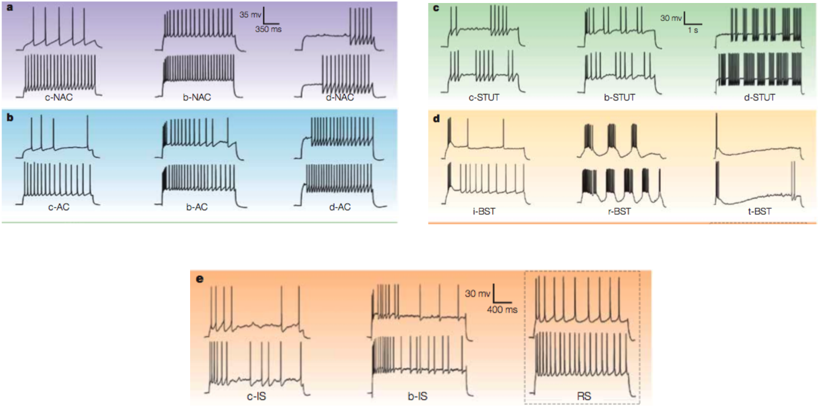




Different spike patterns



Markram *et al* 2004

Phenomenological Neuron Models

Aimed to reproduce the typical behaviors of nerve cells in response to different stimuli:

- Excitability properties of neurons
- Frequency preference property
- subthreshold oscillations,
- Spike patterns fired

Neuron and Dynamical Systems

The main excitability properties can be linked with bifurcations of dynamical systems for

- Continuous dynamical systems: detailed neuron models and their reductions (Rinzel, Ermentrout, Guckenheimer, ...).
- Discrete dynamical systems: map-based models (Caselles, Rulkov, ...)

Hybrid dynamical systems

Integrate-and-fire neuron models combine:

- A continuous dynamical system (ordinary differential equations) accounting for input integration
- A discrete dynamical system (map iteration) accounting for spike emission.

Classical Integrate-and-Fire Neurons

$$\frac{dv}{dt} = -v + I$$
$$v = \theta \Rightarrow \text{Spike!}$$

Louis Lapicque, 1907

Izhikevich (2003)

$$\begin{cases} \dot{v} = v^2 - w + I \\ \dot{w} = a(bv - w) \end{cases}$$

Brette & Gerstner (2005)

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Wehmeier *et al*, 1989

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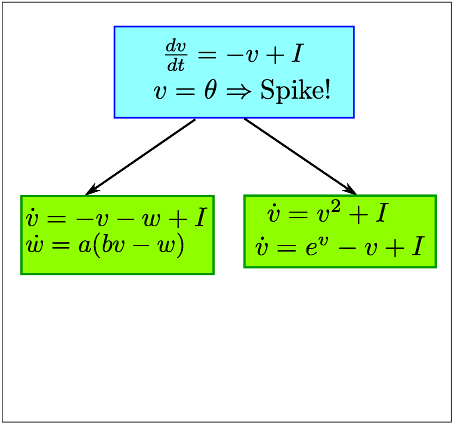

e.g. Ermentrout Kopell, 1982, Fourcaud-Trocme *et al* 2003

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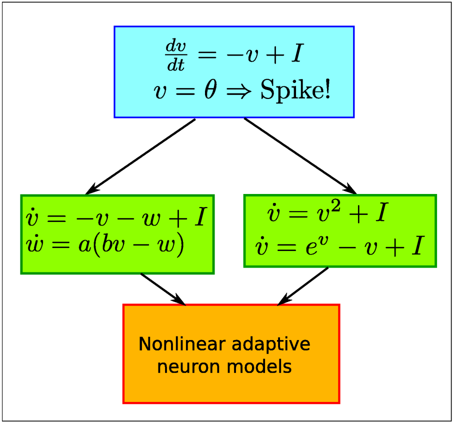

Izhikevich, 2003 & Brette Gerstner 2005

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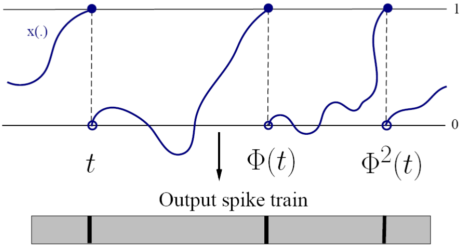
One-dimensional models

$$\dot{x} = F(t,x) \quad F:\mathbb{R}^2 \to \mathbb{R}$$
$$\lim_{s\to t^+} x(s) = 0, \quad \text{if} \quad x(t) = 1$$



Definition [Firing map]

$$\Phi:\mathbb{R}\to\mathbb{R},\ \Phi(t) = \min\{s > t:\ x(s;t,0) = 1\}$$

$$D_\Phi = \{t \in \mathbb{R}: \quad \exists_{s>t}\ x(s;t,0) = 1\}$$

$$t_n = \Phi^n(t) = \min\{s > \Phi^{n-1}(t):\ x(s;\Phi^{n-1}(t),0) = 1\}$$

Perfect Integrator Model

$$\dot{x} = f(t) \qquad \text{(PI)}$$

Leaky Integrate-and-Fire

$$\dot{x} = -\sigma x + f(t) \qquad \text{(LIF)}$$

Non-linear models

$$\dot{x} = F(t, x)$$

- Mathematical analysis of one-dimensional IF models was performed e.g. in [R.Brette, 2004], [H.Carrillo, F.A.Ongay, 2001], [T.Gedeon, M.Holzer, 2004], [W. Marzantowicz, J.S., 2011], [W. Marzantowicz, J.S., 2015] and [P. Kasprzak, A. Nawrocki, J. S., 2015] (with focus on periodic and almost-periodic input functions)

Bidimensional integrate-and-fire models

Bidimensional integrate-and-fire models:

$$\dot{v} = F(v) - w + I \quad (1)$$
$$\dot{w} = a(bv - w) \quad (2)$$

A spike is emitted at time t^* such that

$$\lim_{t \to t^{*-}} v(t) = \infty$$

At the moment of the spike we reset:

$$v(t^{*+}) \longrightarrow v_R, \quad w(t^{*+}) \longrightarrow \gamma w(t^{*-}) + d$$

The adaptation map: $\Phi(w_0) = \gamma w(t^{*-}) + d$, (v_R, w_0)-the initial condition of the solution $(v(t), w(t))$ which spikes at t^*

Examples include *adaptive exponential model* ($F(v) = e^v - v$), *quadratic adaptive model* ($F(v) = v^2$) and *quartic model* ($F(v) = v^4 + 2av$)

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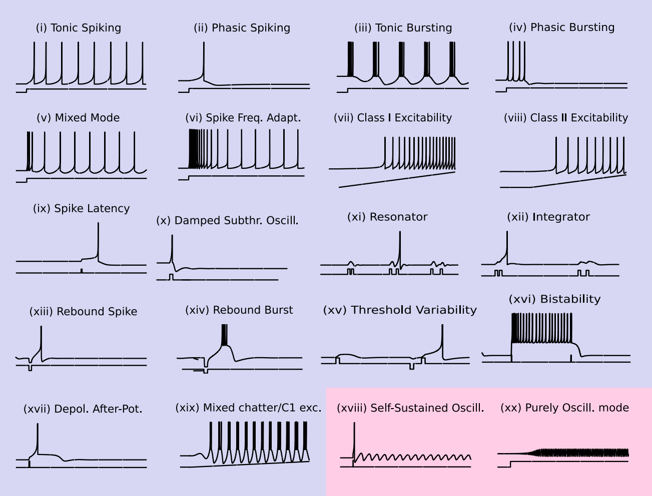
Behaviors of the Quartic Model



The model can display complex dynamics including **Mixed-Mode Oscillations** and **Mixed-Mode Bursting Oscillations** (MM(B)O) that are sequences of spikes interspersed by small subthreshold oscillations.

MM(B)Os so far have been investigated in 3D and higher dimensional systems ([M. Desroches et al., 2012], [M. Krupa et al., 2012], [T. Vo et al., 2012]).

In such hybrid models they have never been observed before.

From the neuroscience point of view, they have been evidenced in Hodgin-Huxley model ([J. Rubin, M.Wechselberger, 2007], [J. Rubin, M.Wechselberger, 2008]) and in the coupled FitzHugh-Nagumo systems ([N. Berglund, D. Landon, 2012], [M. Desroches et al., 2008]).

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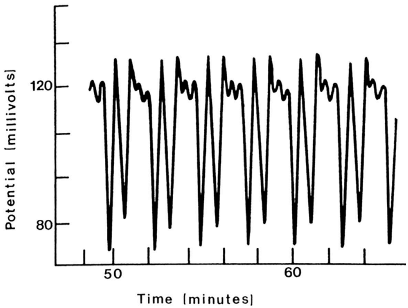
Another (classical) example are chemical reactions:



Bromide ion electrode potential in the Belousov–Zhabotinsky reaction; figure from [J.L. Hudson et al., 1979]

Our aim was to show that they also occur in 2D integrate-and-fire models through the simple geometric mechanism.

[joint work with J. Touboul (Mathematical Neuroscience Lab and EPI MYCENAE) and A. Vidal (LaMME, Univ Evry and EPI MYCENAE)]

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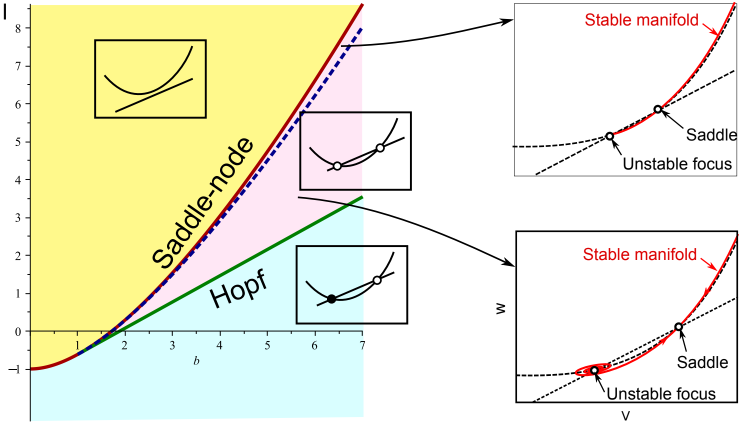
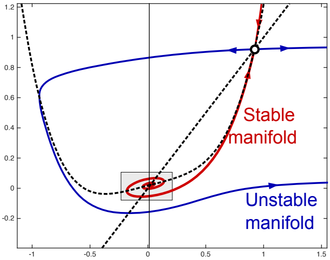
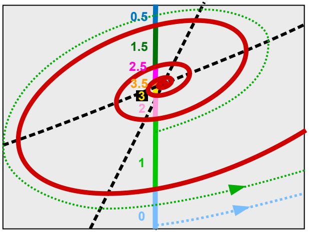
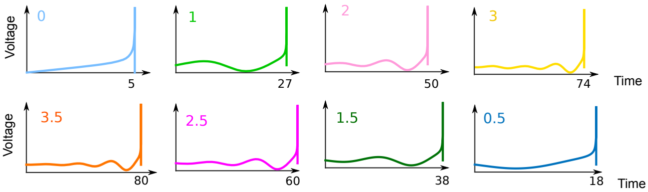

I
8
7
6
5
4
3
2
1
0
−1
1
2
3
4
5
6
7
b
Saddle-node
Hopf
Stable manifold
Saddle
Unstable focus
Stable manifold
Saddle
Unstable focus
w
v

Mixed-Mode Oscillations (MMOs, slow oscillations interspersed with spikes or bursts):







Bidimensional integrate-and-fire models were studied in [R. Brette, W. Gerstner, 2005], [E.Izhikevich, 2003], [N.Jimenez et al., 2013] and [J. Touboul, R.Brette, 2009].

We assume that:

- $F \in C^3(\mathbb{R})$ (at least)
- F is strictly convex
- $\lim_{v\to-\infty} F'(v) < 0$
- there exist $\varepsilon > 0$ and $\delta > 0$ such that:

$$\lim_{v\to\infty} \frac{F(v)}{v^{2+\varepsilon}} \geq \delta$$

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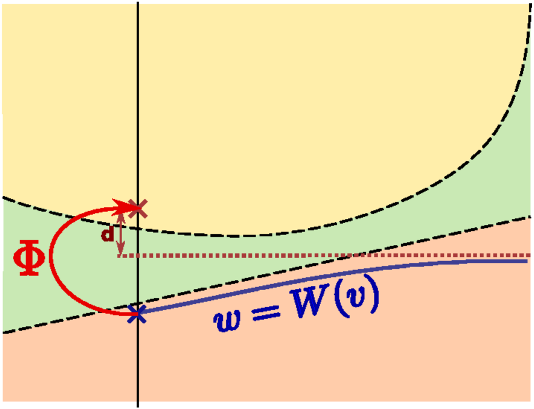
The adaptation map

We define:

- $\mathcal{D}$ the set of w s.t. the solution starting from (v_r, w) spikes.
- $\Phi : \mathcal{D} \mapsto \mathbb{R}$ the function such that $\Phi(w)$ is the after-spike adaptation value.



Definition [Adaptation map]

The adaptation map Φ associates to a value of the adaptation variable w the value of the adaptation variable after reset:

$$\Phi(w) := \gamma W(t^*; v_r, w) + d,$$

where $(V(t; v_r, w), W(t; v_r, w))$ is the solution of the system (1)-(2) with initial condition (v_r, w) at time t, and t^* is the value at which $V(t; v_r, w)$ diverges.

Let $\mathcal{D} = \{w_1, w_2, ...\}$ be the set of intersections of the line $v = v_R$ with SMSFP. Then $\Phi : \mathbb{R} \setminus \mathcal{D} \to \mathbb{R}$ is well-defined.

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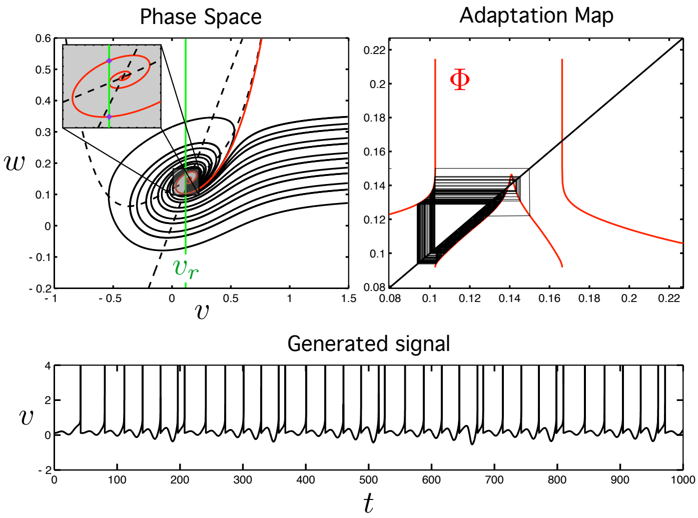
Phase Space
Adaptation Map
Generated signal
w
v
v_r
Φ
t
v
0.6
0.5
0.4
0.3
0.2
0.1
0
- 0.1
- 0.2
- 1
- 0.5
0
0.5
1
1.5
0.22
0.2
0.18
0.16
0.14
0.12
0.1
0.08
0.08
0.1
0.12
0.14
0.16
0.18
0.2
0.22
4
2
0
- 2
0
100
200
300
400
500
600
700
800
900
1000

Remark

Suppose that $(v(t), w(t))$ is the spiking solution starting from (v_r, w_0) and let $\{t_n\}_{n>0}$ be the sequence of spike times for this solution. By $\{w_n\}_{n>0}$ denote the values of the adaptation variable w at spikes, i.e.

$$w_n := w(t_n^+) = \gamma w(t_n^-) + d$$

Then the adaptation map satisfies

$$\Phi(w_n) = w_{n+1}$$

The spike train can be qualitatively described via iterations of Φ, with fixed points of Φ corresponding to tonic, regular spiking and periodic orbits to bursts. Thus the study of the dynamics of Φ allows to discriminate between different spiking patterns.

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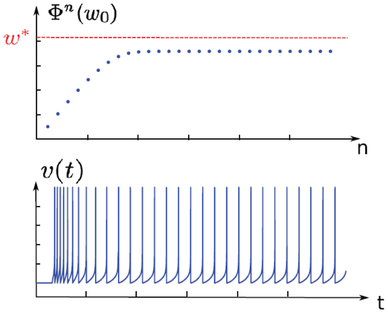
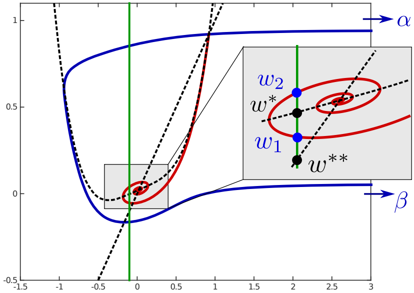


(figures from [J. Touboul, R.Brette, 2009])



- $(w_i)_{i=1\cdots p}$ - intersections of the reset line $\{v = v_r\}$ with SMSFP
- p_1- the index such that $(w_i)_{i\le p_1}$ are below the v-nullcline and $(w_i)_{i>p_1}$ are above
- $(I_i)_{i=0\cdots p+1}$ - intervals with endpoints w_i
- α, β - the value of w after a spike for an initial condition on the upper and, respectively, lower branch of UMSFP

Theorem

The adaptation map has the following properties:

1. *it is defined for all* $w \in \mathcal{D} = \mathbb{R} \setminus \{w_i; i = 1 \cdots p\}$
2. *its regular (at least* C^1*) everywhere except the points* $(w_i)_{i=1\cdots p}$
3. *at the boundaries of the definition domain* $\mathcal{D}$, $\{w_i; i = 1 \cdots p\}$, *the map has well-defined and distinct left and right limits:*

$$\begin{cases} \lim_{w \to w_i^-} \Phi(w) = \alpha, \ \lim_{w \to w_i^+} \Phi(w) = \beta, & i \leq p_1 \\ \lim_{w \to w_j^-} \Phi(w) = \beta, \ \lim_{w \to w_j^+} \Phi(w) = \alpha, & j > p_1 \end{cases}$$

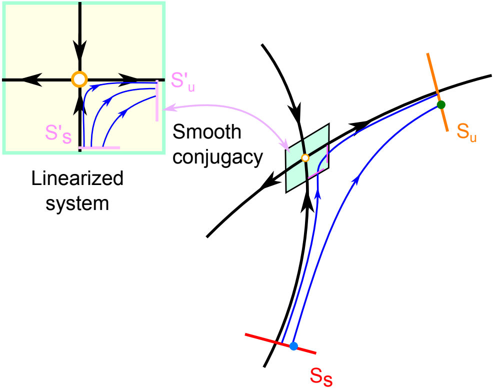
4. *the derivative* $\Phi'(w)$ *diverges at the discontinuity points:*

$$\begin{cases} \lim_{w \to w_i^\pm} \Phi'(w) = \infty & i \leq p_1 \\ \lim_{w \to w_i^\pm} \Phi'(w) = -\infty & i > p_1 \end{cases}$$

- (v) for $w < \min\{\frac{d}{1-\gamma}, w_1, w^{**}\}$ we have $\Phi(w) \geq \gamma w + d > w$
- (vi) $\Phi(w)$ is convex in the left-neighbourhood of w_i (and concave in the right-neighbourhood)
- (vii) $\Phi(w)$ has a horizontal plateau for $w \to \infty$ provided that

$$\lim_{v \to -\infty} F'(v) < -a(b + \sqrt{2})$$

The divergence of the derivative $\lim_{w \to w_1} \Phi'(w) = \infty$ is due to the magnitudes of the eigenvalues $\nu > 0$ and $\mu < 0$ of the saddle fixed point: $|\nu| - \mu > 0$



Assume that the line $v = v_r$ has two intersections with SMSFP: w_1 and w_2, with $w_1 < w_2$. We distinguish the following cases:

I. $\beta < w_1 < \alpha < w_2$

I.′ $\beta < w_1 < w_* < w_2 < \alpha$

I.″ $\beta < \alpha < w_1$

I.‴ $w_1 < \beta < \alpha$

II. $\alpha < w_* < w_2$

II.′ $w_* \leq \alpha < w_2$

III. $\Phi(\beta) \geq \beta$

III.′ $\Phi(\beta) < \beta$

IV.a $\Phi(\alpha) \leq \Phi(\beta)$

IV.b $\Phi(\alpha) > \Phi(\beta)$

V.a $w_1 < w_2 < \beta < \alpha$

V.b $w_1 < \beta < w_2 < \alpha$

V.c $w_1 < \beta < \alpha < w_2$

Assume that the line $v = v_r$ has two intersections with SMSFP: w_1 and w_2, with $w_1 < w_2$. We distinguish the following cases:

I. $\beta < w_1 < \alpha < w_2$

I.′ $\beta < w_1 < w_* < w_2 < \alpha$

I.″ $\beta < \alpha < w_1$

I.‴ $w_1 < \beta < \alpha$

II. $\alpha < w_* < w_2$

II.′ $w_* \leq \alpha < w_2$

III. $\Phi(\beta) \geq \beta$

III.′ $\Phi(\beta) < \beta$

IV.a $\Phi(\alpha) \leq \Phi(\beta)$

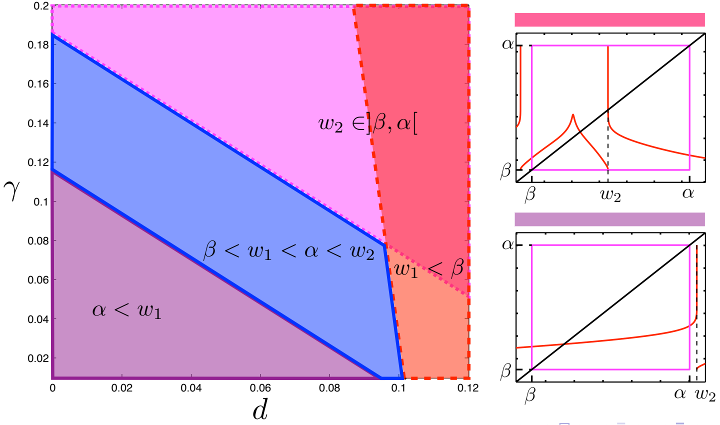
IV.b $\Phi(\alpha) > \Phi(\beta)$

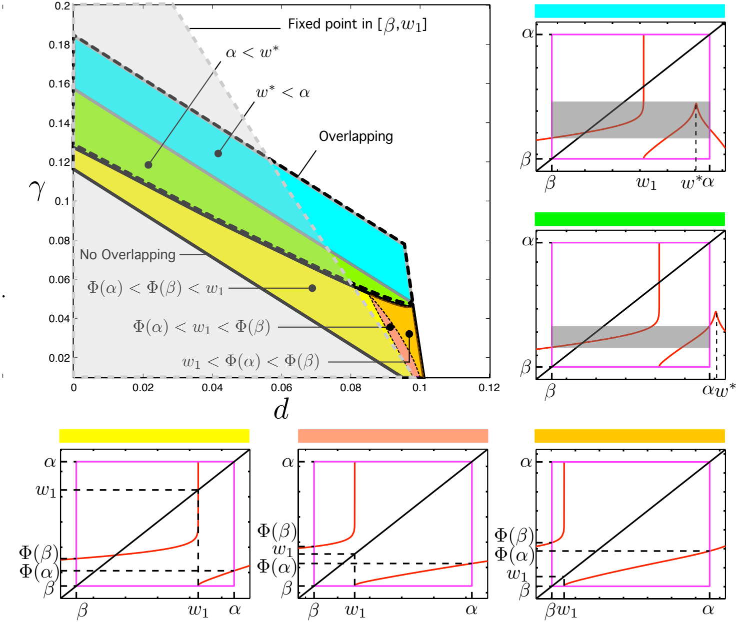
V.a $w_1 < w_2 < \beta < \alpha$

V.b $w_1 < \beta < w_2 < \alpha$

V.c $w_1 < \beta < \alpha < w_2$

Quartic model $F(v) = v^4 + 2cv$ with parameter values: $a = 0.1$, $b = 1$, $c = 0.1$, $I = -3(a/4)^{(4/3)}(2a-1) + 0.1 \approx 0.1175$ and $v_r = 0.1158$




Fixed point in $[\beta, w_1]$
$\alpha < w^*$
$w^* < \alpha$
Overlapping
No Overlapping
$\Phi(\alpha) < \Phi(\beta) < w_1$
$\Phi(\alpha) < w_1 < \Phi(\beta)$
$w_1 < \Phi(\alpha) < \Phi(\beta)$
γ
d
0.2
0.18
0.16
0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
0.02
0.04
0.06
0.08
0.1
0.12
α
β
w_1
w^*
$\Phi(\beta)$
$\Phi(\alpha)$

Firstly, let us assume that

I. $\beta < w_1 < \alpha < w_2$

The above means that the identity line passes through the gap at w_1 and that in the interval $(-\infty, \alpha]$ (where the dynamics concentrates) there is only one discontinuity point w_1.

The analysis of Φ will be cut to the invariant interval $[x, \alpha]$, where $x = \beta$ when $\Phi(\beta) \geq \beta$ or $x = w_f$ when $\Phi(\beta) < \beta$ and $w_f < \beta$ is the greatest fixed point of Φ in $(-\infty, \beta)$

Now let us add the following two assumptions about Φ:

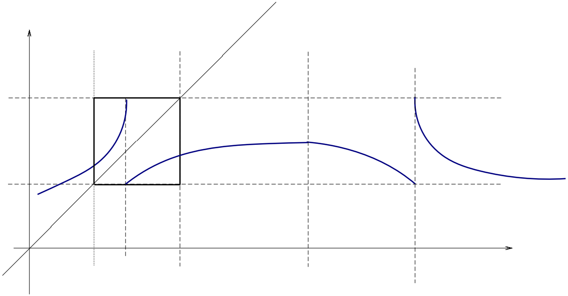
II. $\alpha < w_* < w_2$

III. $\Phi(\beta) \geq \beta$

Proposition

Under I. and II., whenever $\Phi : [x, \alpha] \setminus \{w_1\} \to [x, \alpha]$ has a periodic orbit (with period $q > 1$), this periodic orbit exhibits MMBO. However, this orbit does not need to be stable.

Assume I., II. and III.



We analyze $\Phi : [\beta, \alpha] \to [\beta, \alpha]$:

- $\Phi(w)$ is piecewise C^1 on $[\beta, \alpha]$ with a single jump discontinuity at $w = w_1 \in (\beta, \alpha)$.
- $\lim_{w \to w_1^-} \Phi'(w) = \lim_{w \to w_1^+} \Phi'(w) = \infty$
- $\lim_{w \to w_1^+} \Phi(w) = \beta$ and $\lim_{w \to w_1^-} \Phi(w) = \alpha$

According to [J.P.Keener, 1980] analysis of such maps can be performed separately for the following cases:

(i) non-overlapping case:

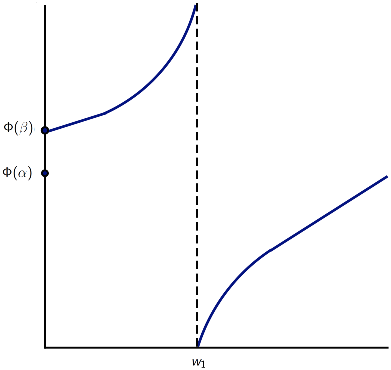
$$\Phi(\alpha) < \Phi(\beta)$$

(ii) overlapping case:

$$\Phi(\alpha) > \Phi(\beta)$$

(iii) $\Phi(\alpha) = \Phi(\beta)$

with the help of the *rotation number*:



$$\varrho(w) := \lim_{n\to\infty} \frac{\Psi^n(w) - w}{n(\alpha - \beta)} \equiv \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \chi_{(w_1,\alpha]}(\Phi^i(w))$$

Non-overlapping case: I., II., III. and IV. a $\Phi(\alpha) < \Phi(\beta)$

We consider the lift $\Psi : \mathbb{R} \to \mathbb{R}$ of Φ by identifying α with β and requiring that $\Psi(w + k(\alpha - \beta)) = \Psi(w) + k(\alpha - \beta)$, for all $k \in \mathbb{Z}$ and $w \in \mathbb{R}$.

Theorem (cf. [R.Brette, 2003] and [F.Rhodes,Ch.Thompson, 1986])

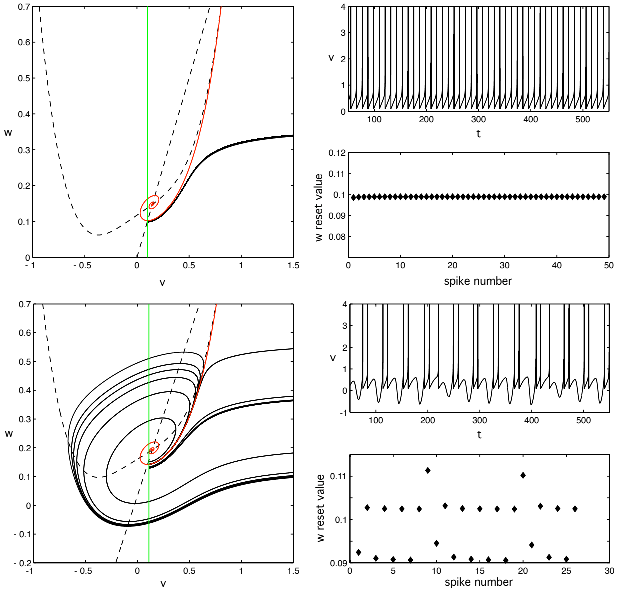
The rotation number

$$\lim_{n\to\infty} \frac{\Psi^n(w) - w}{n(\alpha - \beta)} = \varrho$$

exists and does not depend on $w \in [\alpha, \beta]$.

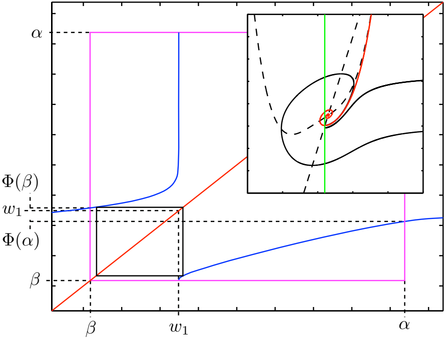
Moreover, if $\varrho = \frac{p}{q} \in \mathbb{Q}$, *then all orbits* $\{\Phi^n(w)\}$, $w \in [\beta, \alpha]$, *tend to a periodic orbit with the same period* q *and if* $\varrho \notin \mathbb{Q}$, *then all orbits have the same limit set which is either the whole* $[\beta, \alpha]$ *or some Cantor subset of it (meaning, in particular there are no periodic orbits).*

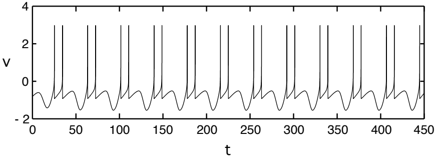
- $\varrho = 0 \mod 1 \implies$ tonic, regular spiking (for every initial condition $w_0 \in [\beta, \alpha] \setminus \{w_1\}$)
- $\varrho = p/q \in \mathbb{Q} \setminus \mathbb{Z} \implies$ MMBO (with periodicity of interspike-intervals and interspersing oscillations)
- $\varrho \in \mathbb{R} \setminus \mathbb{Q} \implies$ no periodic orbits and we observe *chaos*.


w
v
v
t
w reset value
spike number
w
v
v
t
w reset value
spike number

Proposition

Under I., II., III., IV.a., if $\Phi(\beta) > w_1$ and $\Phi(\alpha) < w_1$ then Φ has a periodic orbit of period two, which exhibits MMBO.





If $\Psi : \mathbb{R} \to \mathbb{R}$ is a non-decreasing map of degree-one (i.e. in our case $\Psi(w + (\alpha - \beta)) = \Psi(w) + (\alpha - \beta)$ for every $w \in \mathbb{R}$), then

$$R(\Psi) := \{(x, y) \in \mathbb{R}^2 : \ \Psi^-(x) \leq y \leq \Psi^+(x)\}.$$

Definition [*H*-convergence]

$\Psi_s \xrightarrow{H} \Psi_{s_0}$ as $s \to s_0$ iff $R(\Psi_s) \xrightarrow{R} (\Psi_{s_0})$ in the Hausdorff metric

We say that (Ψ_s) is uniformly convergent to Ψ_{s_0} at x_0 as $s \to s_0$ if for each $\varepsilon > 0$ there exist $\xi > 0$ and $\delta > 0$ such that for all s and x satisfying $|s - s_0| < \xi$ and $|x - x_0| < \delta$ we have $|\Psi_s(x) - \Psi_{s_0}(x_0)| < \varepsilon$. The *H*-convergence for non-decreasing degree-one circle maps can be characterised in a very convenient way:

Proposition (cf. [F.Rhodes,Ch.Thompson, 1991])

If (Ψ_s) is a family of degree-one non-decreasing maps, then $\Psi_s \xrightarrow{H} \Psi_{s_0}$ as $s \to s_0$ if and only if (Ψ_s) is uniformly convergent to Ψ_0 at each point of continuity of Ψ_{s_0}.

If $\Psi : \mathbb{R} \to \mathbb{R}$ is a non-decreasing map of degree-one (i.e. in our case $\Psi(w + (\alpha - \beta)) = \Psi(w) + (\alpha - \beta)$ for every $w \in \mathbb{R}$), then

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If $\Psi : \mathbb{R} \to \mathbb{R}$ is a non-decreasing map of degree-one (i.e. in our case $\Psi(w + (\alpha - \beta)) = \Psi(w) + (\alpha - \beta)$ for every $w \in \mathbb{R}$), then

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Proposition (cf. [F.Rhodes,Ch.Thompson, 1991])

If (Ψ_s) is a family of degree-one non-decreasing maps, then $\Psi_s \xrightarrow{H} \Psi_{s_0}$ as $s \to s_0$ if and only if (Ψ_s) is uniformly convergent to Ψ_0 at each point of continuity of Ψ_{s_0}.

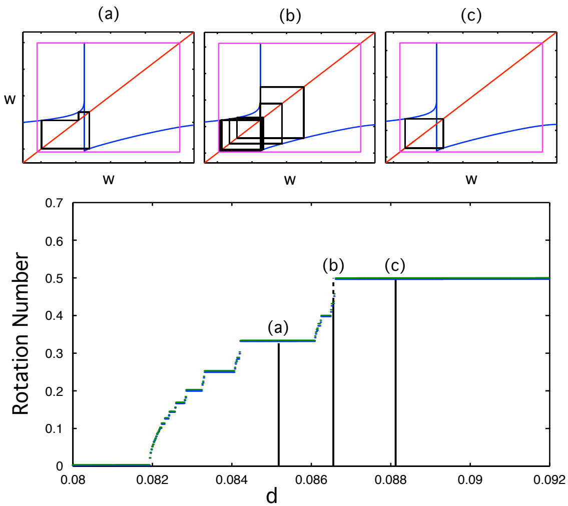
Theorem (cf. [R.Brette, 2003], [F.Rhodes,Ch.Thompson, 1991])

Suppose that $s \mapsto \Phi_s$, $s \in [c,d]$, is a family of adaptation maps with strictly increasing lifts Ψ_s such that the mapping $(s,w) \mapsto \Psi_s(w)$ is increasing with respect to each variable and $s \mapsto \Psi_s$ is continuous with respect to the topology of H-convergence. Let ϱ_s be the rotation number of Ψ_s. Then:

- *$\rho : s \mapsto \varrho_s$ is continuous and non-decreasing;*
- *for all $p/q \in \mathbb{Q} \cap \mathcal{I}m(\rho)$, $\rho^{-1}(p/q)$ is an interval containing more than one point, unless it is $\{c\}$ or $\{d\}$;*
- *ρ reaches every irrational number at most once;*
- *ρ takes irrational values on a Cantor-type subset of $[c,d]$, up to a countable number of points*

Proposition

Let a, b, v_R, I and γ be fixed and consider d varying in some interval $d \in [\lambda_1, \lambda_2]$. Suppose that for this choice of parameter values a, b, v_R, I and γ the adaptation map Φ_d satisfies conditions I., II., III. and IV.a for any value of $d \in [\lambda_1, \lambda_2]$
Let ϱ_d denote the unique rotation number obtained for the map Φ_d (considered on the "fundamental interval" $[\beta_d, \alpha_d]$). Then the mapping $\rho : d \mapsto \varrho_d$ is continuous.
If moreover, for every $d \in [\lambda_1, \lambda_2]$, the adaptation map Φ_d satisfies $\Phi_d(\beta_{\lambda_1}) > \Phi_d(\alpha_{\lambda_2})$, then the above mapping $\rho : d \mapsto \varrho_d$ behaves like a Devil's staircase.



Parameter values: $v_r = 0.1$, $\gamma = 0.05$

- The rotation number $\varrho = p/q \in \mathbb{Q}$ characterises the signature of MM(B)O:

$$\mathcal{L}_1^{s_1}, \mathcal{L}_2^{s_2}, \mathcal{L}_3^{s_3}, \ldots$$

where $\mathcal{L}_i$ denotes the number of big oscillations (spikes) and s_i is the number of following them small su threshold oscillations.

For example, $\varrho = 1/3$ corresponds to the periodic signature 3^1 and $\varrho = 3/5$ to the periodic signature $2^1,\ 1^1,\ 2^1$

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For example, $\varrho = 1/3$ corresponds to the periodic signature 3^1 and $\varrho = 3/5$ to the periodic signature $2^1,\ 1^1,\ 2^1$

Overlapping case: I., II. and IV. b $\Phi(\beta) < \Phi(\alpha)$

The analysis of $\Phi : [\beta, \alpha] \to [\beta, \alpha]$ in the overlapping regime can be made via the results of [M. Misiurewicz, 1986] on *old heavy maps*.

Let $\Psi : \mathbb{R} \to \mathbb{R}$ denote the lift of $\Phi \upharpoonright [\beta, \alpha]$. The map Ψ is a degree one map with only negative jumps. We can define the following:

Definition [Rotation interval $[a(\Psi), b(\Psi)]$]

$$\begin{aligned} a(\Psi) &:= \inf_{w \in \mathbb{R}} \liminf_{n \to \infty} \frac{\Psi^n(w) - w}{n(\alpha - \beta)}, \\ b(\Psi) &:= \sup_{w \in \mathbb{R}} \limsup_{n \to \infty} \frac{\Psi^n(w) - w}{n(\alpha - \beta)}. \end{aligned}$$

An old heavy map does not need to be monotonous in its intervals of continuity and therefore:

Remark

If we assume IV.b, then we can skip the assumption II. since the induced lift Ψ remains an old heavy map.

Overlapping case: I., II. and IV. b $\Phi(\beta) < \Phi(\alpha)$

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$$
\begin{aligned}
a(\Psi) &:= \inf_{w\in\mathbb{R}} \liminf_{n\to\infty} \frac{\Psi^n(w) - w}{n(\alpha - \beta)}, \\
b(\Psi) &:= \sup_{w\in\mathbb{R}} \limsup_{n\to\infty} \frac{\Psi^n(w) - w}{n(\alpha - \beta)}.
\end{aligned}
$$

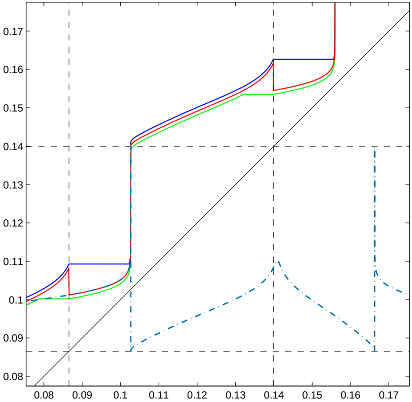
An old heavy map does not need to be monotonous in its intervals of continuity and therefore:

Remark

If we assume IV.b, then we can skip the assumption II. since the induced lift Ψ remains an old heavy map.

Define:

$$\Psi_l(w) := \inf\{\Psi(z) : z \geq w\}$$
$$\Psi_r(w) := \sup\{\Psi(z) : z \leq w$$



The maps $\Psi_l(w)$ and $\Psi_r(w)$ are continuous and non-decreasing thus they admit unique rotation numbers.

The nontrivial rotation interval corresponds to complex dynamics (cf. [M. Misiurewicz, 1986]):

a) if Φ has a q-periodic point w with the rotation number $\varrho(\Psi, w) = p/q$, then $a(\Psi) \leq p/q \leq b(\Psi)$;
b) if $a(\Psi) < p/q < b(\Psi)$, then Φ has a periodic point w of period q and the rotation number $\varrho(\Psi, w) = p/q$

The coexistence of periodic orbits with infinitely many different periods (non-triviality of the rotation interval) is also sometimes called *chaos* (see [J.P.Keener, 1980]).

Remark

If we additionally assume II., i.e. the monotonicity of Φ in the continuity intervals $[\beta, w_1)$ and $(w_1, \alpha]$, then every such a periodic orbit exhibits MMBO (with both one and no small oscillations between consecutive spikes).

Theorem (cf. [M. Misiurewicz, 1986])

Suppose that the adaptation map $\Phi : [\beta, \alpha]$ is in the overlapping case and that for some ϱ_1 and ϱ_2 we have $a(\Psi) \leq \varrho_1 \leq \varrho_2 \leq b(\Psi)$. Then there exists w_0 such that

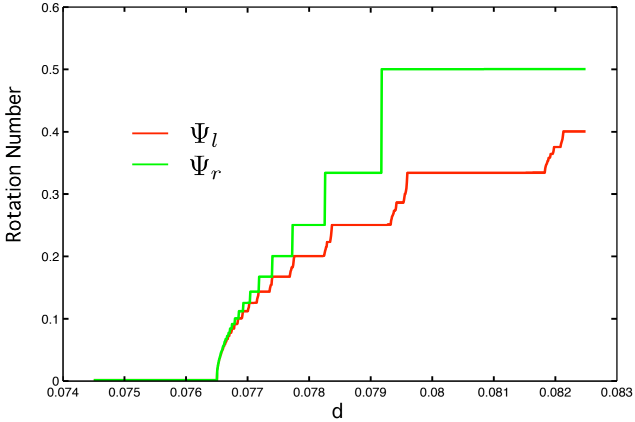
$$\liminf_{n\to\infty} \frac{\Psi^n(w_0) - w_0}{n(\alpha - \beta)} := \varrho_1$$

$$\limsup_{n\to\infty} \frac{\Psi^n(w_0) - w_0}{n(\alpha - \beta)} := \varrho_2$$

Proposition

Choose the fixed parameters v_R, a, b, γ and I and the parameter $d \in [\lambda_1, \lambda_2]$ such that for each $d \in [\lambda_1, \lambda_2]$ the map Φ_d is in the overlapping case, i.e. satisfies I., III. and IV.b. Then the maps $d \mapsto a(\Psi_d)$ and $d \mapsto b(\Psi_d)$, assigning to d the endpoints of the rotation interval of Φ_d, are continuous.

Moreover, usually the maps $d \mapsto a(\Psi_d)$ and $d \mapsto b(\Psi_d)$ also behave as Devil's staircase:



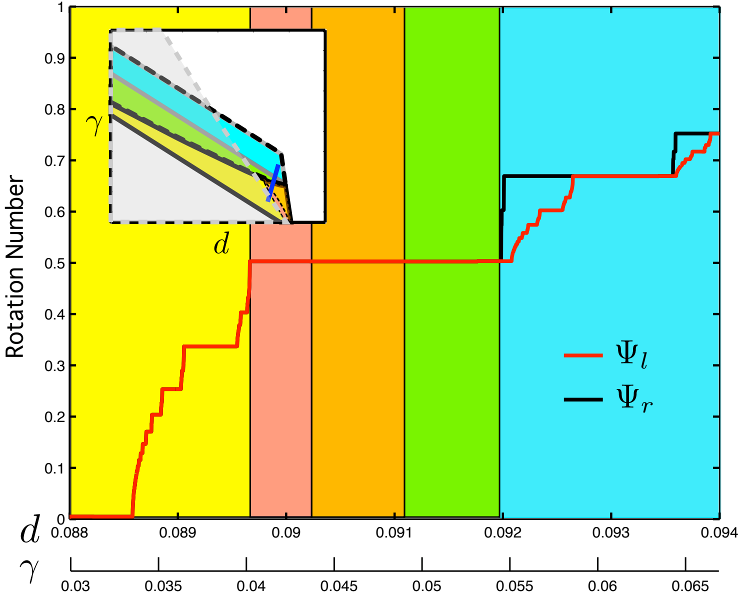
Parameter values: $v_r = 0.1$, $\gamma = 0.05$

Theorem (Chaos)

Suppose that Φ satisfies I., II., III and IV. b (an overlapping case with additional monotonicity condition II.). Further assume also that $\Phi(\alpha) < w_1$ and that Φ has at least two periodic orbits, one with period q_1 and the other with period $q_2 \neq q_1$ and that ***exactly one*** *point of each of these periodic orbits is greater than w_1. Then the mapping $w \mapsto \Phi(w)$ is a* ***shift on a sequence space****.*

Theorem (Condition for orbits of all periods.)

Existence of a fixed point $w_f \in (\beta, w_1)$ and a periodic orbit with period $q > 1$ implies existence of periodic orbits with arbitrary periods $\tilde{q} > q$ and with MMBO. The same holds if $w_f \in (w_1, \alpha)$ provided that the q-periodic orbit is not of the type q/q (i.e. it admits points to the left and to the right of w_1).
In particular, whenever there is a fixed point $w_f \in (\beta, \alpha)$ and a periodic orbit of the type $1/2$, then there are periodic orbits of all periods, exhibiting MMBO.


Rotation Number
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
d
0.088
0.089
0.09
0.091
0.092
0.093
0.094
γ
0.03
0.035
0.04
0.045
0.05
0.055
0.06
0.065
Ψ_l
Ψ_r

Case of both positive and negative jumps: $\{$ I., II. and III.$'$ $\}$ or $\{$ I., II.$'$ and III.$'\}$

I. $\beta < w_1 < \alpha < w_2$

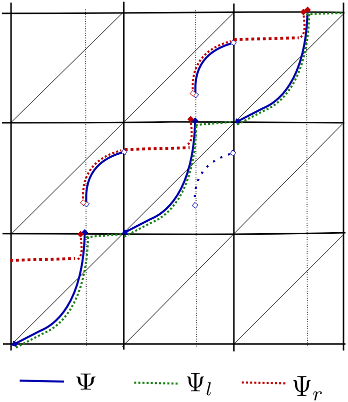
II. $\alpha < w_* < w_2$

II.$'$ $w_* \leq \alpha < w_2$

III.$'$ $\Phi(\beta) < \beta$

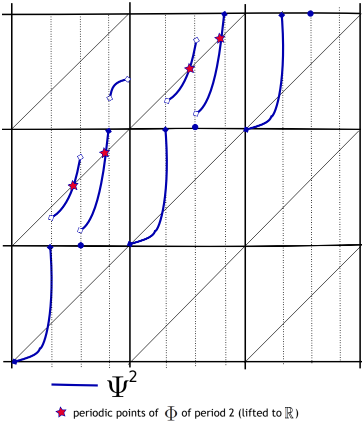
Upper estimate
of the rotation interval:

$$[\varrho(\Psi_l), \varrho(\Psi_r)] \supset [a(\Psi), b(\Psi)]$$



Proposition

Under the assumptions { I., II. and III.′ } or { I., II.′ and III.′ }, if $\Phi(\alpha) > w_1$ and there are no fixed points in (w_1, α), then Φ has an unstable periodic orbit of period 2. This orbit exhibits MMBO but it is unstable.



No discontinuity points in the invariant interval: *I*.″
Under the following condition

I.″ $\beta < \alpha < w_1$

there are no discontinuity points w_1 or w_2 in the interval $(-\infty, \alpha)$.
This is the easiest situation:

- since $\Phi(w) > w$ for $w < \min\{\frac{d}{1-\alpha}, w_1, w_{**}\}$ there must be a fixed point in $(-\infty, \alpha)$ and every point w tends under Φ to one of the fixed points. Thus here we observe for every initial condition **tonic, regular spiking** (in particular, we have no MMO and MMBO) and the dynamics is very simple.

No discontinuity points in the invariant interval and the identity line passes below the gap at w_1: I.‴

I.‴ $w_1 < \beta < \alpha$

V.a $w_1 < w_2 < \beta < \alpha$

Theorem

Suppose that $\lim_{w\to\infty} \Phi(w) > w_2$. Then every point $w > w_1$ is forward asymptotic either to the fixed point $w_{f,1}$ or to a period two orbit. Under these assumptions no point $w \in \mathbb{R}$ exhibits MMO or MMBO.

No discontinuity points in the invariant interval and the identity line passes below the gap at w_1: I.'''

I.''' $w_1 < \beta < \alpha$

V.b $w_1 < \beta < w_2 < \alpha$

Theorem

If $\Phi(w^*) < w_2$, then for every $w \in (w_1, w_2)$ we have $\omega(w) \subset \bar{P}$, where $\bar{P}$ denotes the closure of the set of periodic points of $\Phi : [w_1, w_2] \to [w_1, w_2]$. Particularly, if the set P is finite, then every w tends to some periodic orbit (or fixed point) (with no MM(B)O).

No discontinuity points in the invariant interval and the identity line passes below the gap at w_1: I.‴

I.‴ $w_1 < \beta < \alpha$

V.c $w_1 < \beta < \alpha < w_2$

Theorem

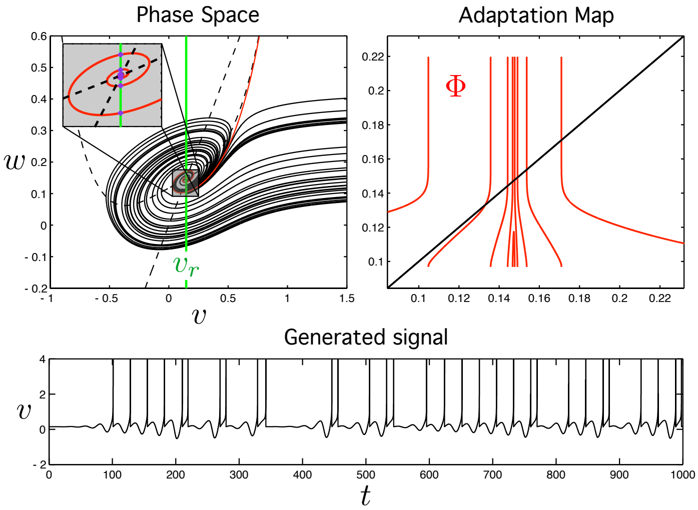
Suppose that $\beta < w^* < \alpha$. If

$$\min \Phi^{-2}(w_*) < \Phi^2(w_*) < \min \Phi^{-1}(w_*) < w_* < \Phi(w_*)$$

and $\Phi(w) > w$ for $w \in (\beta, w_*)$, then $\Phi : [\beta, \alpha] \to [\beta, \alpha]$ has an orbit of period 3. Consequently, Φ has cycles of any period. However, these periodic orbits do not present MMBO.

Conclusions:

- We are able to predict the output properties using geometrical analysis
- In the overlapping and non-overlapping cases existing mathematical tools of rotation theory provide complete description of the dynamics of Φ
- In the remaining cases (e.g. of both positive and negative jumps) one can obtain weaker results on the dynamics of Φ; in particular the rotation interval computed via the enveloping maps Ψ_l and Ψ_r gives the upper-estimate for the possible types p/q of periodic orbits


Phase Space
Adaptation Map
w
v
v_r
Φ
Generated signal
t
0.6
0.5
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-1
-0.5
0
0.5
1
1.5
0.22
0.2
0.18
0.16
0.14
0.12
0.1
0.1
0.12
0.14
0.16
0.18
0.2
0.22
4
2
0
-2
0
100
200
300
400
500
600
700
800
900
1000

Perspectives:

- For multiple discontinuity points the dynamics is even more complex and harder to be completely classified. However, some rigorous results can be obtained via the theory of piece-wise continuous piece-wise monotone maps.
- Consider forcing of the IF system through variable I. A simple starting point is a square signal for $I(t)$: the performed analysis can be generalized using a stroboscopic map.
- Tackle the problematic of 3D vector field appearing with two recovery variables. In this case we have $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$. The general mechanism for generating MMBO is the same, yet leading to richer behaviors due to the geometric structure of the flow.

Thank you!

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